Exercise 13

Prove the identity.

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

Solution

Use the definitions listed on page 259.

$$\begin{aligned}
\coth^2 x - 1 &= \left(\frac{\cosh x}{\sinh x}\right)^2 - 1 \\
&= \frac{\cosh^2 x}{\sinh^2 x} - 1 \\
&= \frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} \\
&= \frac{\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2}{\sinh^2 x} \\
&= \frac{\left(\frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\right)}{\sinh^2 x} \\
&= \frac{\cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} + \cancel{\frac{1}{4}e^{-2x}} - \cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} - \cancel{\frac{1}{4}e^{-2x}}}{\sinh^2 x} \\
&= \frac{1}{\sinh^2 x} \\
&= \operatorname{csch}^2 x
\end{aligned}$$